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Conditional entropies and their relation to entanglement criteria

Karl Gerd H. Vollbrecht* and Michael M. Wolf†
Institute for Mathematical Physics, TU Braunschweig, Germany
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We discuss conditional Rényi and Tsallis entropies for bipartite quantum systems of finite dimension. We investigate the relation between the positivity of conditional entropies and entanglement properties. It is in particular shown that any state having a negative conditional entropy with respect to any value of the entropic parameter is distillable since it violates the reduction criterion. Moreover we show that the entanglement of Werner states in odd dimensions can neither be detected by entropic criteria nor by any other spectral criterion.

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I. INTRODUCTION

Entanglement has always been a key issue in the ongoing debate about the foundations and interpretation of quantum mechanics since Einstein [1] and Schrödinger [2] expressed their deep dissatisfaction about this astonishing part of quantum theory. Whereas for the long period from 1935 to 1964, until Bell [3] published his famous work, discussions about entanglement were purely metatheoretical, nowadays quantum information theory has established entanglement as a physical resource and key ingredient for quantum computation and quantum information processing. This led to a dramatic increase of general structural knowledge about entanglement in the last few years, and the resource point of view often led to results that are reminiscent of those known from thermodynamics: *free entanglement* is distinguished from *bound entanglement* [4], irreversibility can be observed in the process of preparing and distilling entangled states [5] and entanglement itself is defined in a way that it must not increase by means of local operations and classical communication (LOCC). Moreover, there is recent effort in order to quantify quantum correlations through heat engines [6].

Entropies lay at the heart of both theories, thermodynamics and entanglement theory. Concerning the latter it was shown that few reasonable assumptions lead to a unique measure of entanglement [7] for pure bipartite quantum states, which is just the *von Neumann entropy* of the reduced state. Hence, it is obvious that the two subsystems of a pure entangled state exhibit more disorder as the system as a whole, such that the respective conditional entropy is negative. This is a remarkable property of entangled states, which is impossible for classical systems (i.e. classical random variables).

The present paper is primarily devoted to settling the relationship between the negativity of conditional Rényi and Tsallis entropies and other entanglement properties. We will in particular show how the property of having a positive conditional entropy enters into the known implication chain of entanglement resp. separability criteria.

In the second part we will then follow the result of Nielsen and Kempe [8] and give examples of entangled states having the property that their entanglement can neither be detected by entropic criteria nor by any other spectral criterion. Sec.IV shows that this is indeed the case for symmetric *Werner states* (in odd dimensions), which play a crucial role in entanglement theory.

*Electronic address: k.vollbrecht@tu-bs.de
†Electronic address: mm.wolf@tu-bs.de

II. PRELIMINARIES ON SEPARABILITY CRITERIA

To fix ideas we will start by recalling some of the basic notions and previous results concerning separability resp. entanglement criteria.

A bipartite quantum state described by its density matrix ρ acting on a Hilbert space $\mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$ is said to be *separable*, unentangled or classically correlated if it can be written as a convex combination of tensor product states [9]

$$\rho = \sum_j p_j \rho_j^{(A)} \otimes \rho_j^{(B)}, \tag{1}$$

where the positive weights p_j sum up to one and $\rho^{(A)}$ ($\rho^{(B)}$) describes a state on $\mathcal{H}^{(A)}$ ($\mathcal{H}^{(B)}$). This means in particular that pure states are separable if and only if they are product states. Moreover, all entanglement properties of pure states, which can always be written in their Schmidt form (cf. [10]) as $|\Psi\rangle = \sum_i \sqrt{\lambda_i}|i\rangle \otimes |i\rangle$, are completely determined by the eigenvalues $\{\lambda_i\}$ of the reduced state $\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$. The unique measure of entanglement for pure states is then given by the *von Neumann entropy* of the reduced state:

$$S_1(\rho_A) = -\mathrm{tr}\big(\rho_A \log \rho_A\big). \tag{2}$$

For mixed quantum states however, the situation is much more difficult and deciding whether a state is entangled or separable is not yet feasible in general. Currently, the most efficient necessary criterion for separability is the positivity of the partial transpose (PPT), i.e., the condition that ρ^{T_A} has to be a positive semi-definite operator [11]. The partial transpose of the state is thereby defined in terms of its matrix elements with respect to some basis by $\langle kl|\rho^{T_A}|mn\rangle = \langle ml|\rho|kn\rangle$. For the smallest non-trivial systems with 2×2 resp. 2×3 dimensional Hilbert spaces and a few other special cases the PPT-criterion also turned out to be sufficient [12]. In higher dimensional systems, however, so called *bound entangled* states exist, which satisfy the PPT-condition without being separable [4].

Another well known condition is given by the *reduction criterion* [13, 14]

$$\rho_A \otimes \mathbf{1} - \rho \geq 0, \quad \text{and} \quad \mathbf{1} \otimes \rho_B - \rho \geq 0, \tag{3}$$

which is implied by the PPT-criterion but nevertheless an important condition since its violation implies the possibility of recovering entanglement by distillation (which is yet unclear for PPT violating states). For the case of two qubits (and 2×3) the reduction criterion is also known to be sufficient for separability [13, 14]. Moreover, it was shown in [15] that Eq.(3) implies that the rank of the reduced state has to be smaller or equal than the rank of ρ. The general line of implication is then:

$$\begin{array}{c} \rho \text{ separable} \\ \Downarrow \\ \rho^{T_A} \geq 0 \\ \Downarrow \\ \rho \text{ undistillable} \\ \Downarrow \\ \rho_A \otimes \mathbf{1} - \rho \geq 0 \quad \wedge \quad \mathbf{1} \otimes \rho_B - \rho \geq 0 \\ \Downarrow \\ \max\left[\operatorname{rank}(\rho_A), \operatorname{rank}(\rho_B)\right] \leq \operatorname{rank}(\rho) \end{array} \tag{4}$$

The last condition we want to mention was recently derived by Nielsen and Kempe [8] and is based on majorization. However, it is yet not known how the *majorization criterion* enters into the above implication chain. Since it is closely related to conditional entropies we will discuss it in more detail in the following section.

III. CONDITIONAL ENTROPIES

The idea to use entropic inequalities as separability resp. entanglement criteria for mixed states goes back to the mid nineties when Cerf and Adami [16] and the Horodecki family [17] recognized that certain conditional Rényi entropies are non-negative for separable states, and it was recently resurrected by several groups [18, 19, 20, 21, 22, 23] in the form of conditional Tsallis entropies.

The quantum *Rényi entropy* depending on the entropic parameter $\alpha \in \mathbf{R}$ is given by

$$S_\alpha(\rho) = \frac{\log \operatorname{tr}(\rho^\alpha)}{1-\alpha}, \tag{5}$$

where S_0, S_1, S_∞ reduces to the logarithm of the rank, the von Neumann entropy and the negative logarithm of the operator norm respectively. For the case of separable states it was shown in [15, 16, 17] that the conditional entropy [24]:

$$S_\alpha(B|A;\rho) := S_\alpha(\rho) - S_\alpha(\rho_A) \tag{6}$$

is non-negative for $\alpha = 0, \infty$ and $\alpha \in [1,2]$.

In Ref. [18, 20] essentially the same criterion was expressed in terms of the *Tsallis entropy*

$$T_\alpha(\rho) = \frac{1-\operatorname{tr}(\rho^\alpha)}{\alpha - 1}, \tag{7}$$

which is non-negative, concave (convex) for $\alpha > 0$ ($\alpha < 0$) and becomes the von Neumann entropy in the limit $\alpha \to 1$. The *conditional Tsallis entropy* defined in [18] reads

$$T_\alpha(B|A;\rho) := \frac{\operatorname{tr}(\rho_A^\alpha) - \operatorname{tr}(\rho^\alpha)}{(\alpha-1)\operatorname{tr}(\rho_A^\alpha)}\,. \tag{8}$$

Concerning positivity, however, the two conditional entropies are equivalent, i.e.

$$T_\alpha(B|A;\rho) \geq 0 \quad \Leftrightarrow \quad S_\alpha(B|A;\rho) \geq 0, \tag{9}$$

which is in turn equivalent to $\operatorname{tr}(\rho_A^\alpha) \geq \operatorname{tr}(\rho^\alpha)$ for $\alpha > 1$, $\operatorname{tr}(\rho_A^\alpha) \leq \operatorname{tr}(\rho^\alpha)$ for $0 \leq \alpha < 1$, and the positivity of the conditional von Neumann entropy for $\alpha = 1$.

Obviously, for pure states the conditional entropies are negative if and only if the state is entangled.

A. Monotonicity counterexample

It was conjectured in [20] that $T_\alpha(B|A;\rho)$ is monotonically decreasing in α, such that it would be sufficient to calculate $T_\infty(B|A;\rho)$ in order to decide positivity. However, monotonicity does not hold in general and can most easily be ruled out by low rank examples like

$$\rho = \frac{1}{2}\big(|\Phi_+\rangle\langle\Phi_+| + |01\rangle\langle 01|\big), \quad |\Phi_+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big),$$

for which the reduced state has eigenvalues $\frac{1}{4}, \frac{3}{4}$ and therefore $T_0 = T_\infty = 0 \neq T_2 = \frac{1}{5}$. We note that similar counterexamples can be found for the monotonicity of the conditional Rényi entropy as well. Fortunately, however, monotonicity is not necessary for proving the positivity of the conditional Tsallis/Rényi entropies for separable states for other values than $\alpha = 0, \infty, \alpha \in [1,2]$ [25].

B. Majorization and convex functions

Majorization turned out to be a powerful tool in the discussion of quantum state transformations by means of LOCC operations (cf.[26]) and it was recently proven to yield the strongest separability criterion, which is based on the spectra of a state and one of its reductions. It was proven in Ref. [8] that any separable state ρ acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ is majorized by its reduced state ρ_A:

$$\rho_A \succ \rho \quad \text{i.e.} \quad \forall k \leq d : \sum_{i=1}^{k} \lambda_i^{(A)} \geq \sum_{i=1}^{k} \lambda_i, \tag{10}$$

where $\{\lambda_i\}$ and $\{\lambda_i^{(A)}\}$ are the decreasingly ordered eigenvalues of ρ respectively ρ_A.

It is a well known result in the theory of majorization that $x \succ y$ iff $\mathrm{tr}\big(f(x)\big) \geq \mathrm{tr}\big(f(y)\big)$ for all convex functions $f : \mathbb{R} \to \mathbb{R}$ [27]. Since $f(x) = x^\alpha$ is convex for $\alpha \geq 1$, concave on $\mathbb{R}^+$ for $0 \leq \alpha \leq 1$ and the von Neumann entropy is concave (needed for $\alpha = 1$), this immediately implies:

Theorem 1 *Let ρ be a bipartite quantum state, which is majorized by its reduction $\rho_A \succ \rho$, then for every $\alpha \geq 0$ the conditional Tsallis/Rényi entropies of ρ are non-negative, i.e.:*

$$S_\alpha(B|A;\rho) \geq 0 \quad \textit{and} \quad T_\alpha(B|A;\rho) \geq 0. \tag{11}$$

The result of Nielsen and Kempe implies that this holds in particular for any separable state.

It is yet not known how the majorization criterion (10) is related to other separability criteria like PPT, undistillability and the reduction criterion. However, we will show in the next subsection how the positivity of conditional entropies is related to these properties.

C. Conditional Entropies and the Reduction Criterion

Positivity of the conditional entropies for $\alpha = 0$ reduces to the rank criterion in the implication chain (4). The following theorem will show, however, that all the other properties stated in (4) in turn imply positivity of the conditional entropies for every value of the entropic parameter α.

Theorem 2 *Let ρ be a bipartite quantum state satisfying the reduction criterion $\rho_A \otimes \mathbf{1} \geq \rho$. Then for every $\alpha \geq 0$ the conditional Tsallis/Rényi entropies are non-negative:*

$$S_\alpha(B|A;\rho) \geq 0 \quad \textit{and} \quad T_\alpha(B|A;\rho) \geq 0. \tag{12}$$

We note that Thm.2 implies in particular, that states with negative conditional entropies are distillable.

Proof: We will divide the proof into three steps depending on the value of the entropic parameter.

For $\alpha > 1$ the proof is essentially based on the Golden-Thompson inequality (cf.[28]) stating that

$$\mathrm{tr}\big(e^A e^B\big) \geq \mathrm{tr}\big(e^{A+B}\big) \tag{13}$$

for hermitian matrices A, B. Utilizing the definition of the reduced state, i.e.,

$$\forall P \geq 0 : \mathrm{tr}\big(\rho(P \otimes \mathbf{1})\big) \equiv \mathrm{tr}\big(\rho_A P\big) \tag{14}$$

this leads to:

$$\begin{aligned} \mathrm{tr}\big(\rho_A^\alpha\big) &= \mathrm{tr}\big[\rho(\rho_A^{\alpha-1} \otimes \mathbf{1})\big] \\ &= \mathrm{tr}\Big[\exp\big(\ln\rho\big)\exp\big((\alpha-1)\ln(\rho_A \otimes \mathbf{1})\big)\Big] \\ &\geq \mathrm{tr}\Big[\exp\big(\ln(\rho) + (\alpha-1)\ln(\rho_A \otimes \mathbf{1})\big)\Big] \end{aligned} \tag{15}$$

At this point we need two monotonicity properties in order to exploit the validity of the reduction criterion. First of all we use the fact that the logarithm is operator monotone [29], i.e.

$$A \geq B \Rightarrow \ln A \geq \ln B. \tag{16}$$

Thus, for $\alpha > 1$ the reductions criterion $\rho_A \otimes \mathbf{1} \geq \rho$ implies

$$\begin{aligned} \ln(\rho) + (\alpha-1)\ln(\rho_A \otimes \mathbf{1}) &\geq \ln(\rho) + (\alpha-1)\ln(\rho) \\ &= \alpha\ln(\rho). \end{aligned} \tag{17}$$

In the second step we utilize the fact that the exponential function is monotone under the trace. This can be seen by noting that for any A hermitian, $P \geq 0$ and $B = (A + \epsilon P)$ with $\epsilon \geq 0$:

$$\frac{\partial}{\partial\epsilon}\mathrm{tr}\big(e^B\big) = \mathrm{tr}\big(e^B P\big) \geq 0. \tag{18}$$

Hence $\mathrm{tr}\big(e^B\big) \geq \mathrm{tr}\big(e^A\big)$ is implied by $B \geq A$. Together with Eq. (17) this leads to:

$$\mathrm{tr}\big(\rho_A^\alpha\big) \geq (15) \;\geq\; \mathrm{tr}\Big[\exp\big(\alpha\ln\rho\big)\Big] = \mathrm{tr}\big(\rho^\alpha\big). \tag{19}$$

For $0 \leq \alpha < 1$ the reduction criterion can immediately be applied since $f(A) = A^r$ is an operator decreasing function for $-1 \leq r \leq 0$, $A \geq 0$ (cf.[30]) and thus

$$\mathrm{tr}\big(\rho_A^\alpha\big) = \mathrm{tr}\big[\rho(\rho_A^{\alpha-1} \otimes \mathbf{1})\big] \leq \mathrm{tr}\big(\rho^\alpha\big). \tag{20}$$

For the case $\alpha = 1$ we hav to look at the conditional von Neumann entropy $S_1(\rho) - S_1(\rho_A)$, for which positivity is directly implied by the reduction criterion and the operator monotonicity of the logarithm:

$$\begin{aligned} S_1(\rho_A) &= -\mathrm{tr}\rho_A \log\rho_A &(21)\\ &= -\mathrm{tr}\rho\log\rho_A \otimes \mathbf{1} &(22)\\ &\leq -\mathrm{tr}\rho\log\rho &(23)\\ &= S_1(\rho), &(24) \end{aligned}$$

which completes the proof.

$\square$

D. Negative entropic parameters

So far we have only discussed conditional entropies for non-negative values of the entropic parameter α. For these cases we know that they can become negative for entangled states, the simplest examples being pure entangled states. However, for $\alpha < 0$ (and states of full rank) the sign of the conditional entropy contains no information:

Theorem 3 *Let ρ be a bipartite quantum state of full rank. Then for every $\alpha < 0$ the conditional Tsallis/Rényi entropies are non-negative:*

$$\forall\alpha < 0: \; S_\alpha(B|A;\rho) \geq 0 \quad \textit{and} \quad T_\alpha(B|A;\rho) \geq 0. \tag{25}$$

Proof: Let $\{|a\rangle\}$ be an eigenbasis of ρ_A. Then:

$$\begin{aligned} \mathrm{tr}\big(\rho_A^\alpha\big) &= \sum_a \langle a|\rho_A|a\rangle^\alpha & (26)\\ &= \sum_a \Big[\sum_i \langle a\otimes i|\rho|a\otimes i\rangle\Big]^\alpha & (27)\\ &\leq \sum_{a,i} \langle a\otimes i|\rho|a\otimes i\rangle^\alpha \leq \mathrm{tr}\big(\rho^\alpha\big), & (28)\end{aligned}$$

where Eq.(27-28) uses that $\big(\sum_i b_i\big)^\alpha \leq \sum_i b_i^\alpha$ holds for $b_i \geq 0$, $\alpha \leq 0$, and the last inequality is implied by the convexity of negative powers on $\mathbf{R}^+$.

$\square$

IV. ISOSPECTRAL STATES

The fact that positivity of conditional entropies is implied by the reduction criterion (Thm.2) shows already that such an entropic criterion cannot be sufficient for separability. In fact, it was shown in Ref. [8] that no spectral property is capable of distinguishing any entangled state from separable ones.

We will in this section follow the idea of Ref. [8] and construct particular examples of states, such that their entanglement cannot be detected by any spectral criterion, since there exist separable states having the same spectrum and the same reductions.

Werner states [9] have always played an important and paradigmatic role in quantum information theory. Their characteristic property is that they commute with all unitaries of the form $U\otimes U$ and they can be expressed as

$$\rho(p) = (1-p)\frac{P_+}{r_+} + p\frac{P_-}{r_-}, \quad 0 \leq p \leq 1, \tag{29}$$

where P_+ (P_-) is the projector onto the symmetric (antisymmetric) subspace of $\mathbb{C}^d\otimes\mathbb{C}^d$ and $r_\pm = \mathrm{tr}[P_\pm] = \frac{d^2\pm d}{2}$ are the respective dimensions. Werner showed that these states are entangled iff $p > \frac{1}{2}$ independent of the dimension d. The following shows however, that none of these entangled states for odd dimension d can be detected by any separability criterion, which is based on the spectrum of the state and its reductions.

Theorem 4 *Any entangled state in $\mathbb{C}^d\otimes\mathbb{C}^d$ with maximal chaotic reductions and eigenvalues having multiplicities which are multiples of d, has a separable isospectral counterpart, which is locally undistinguishable as it has the same reductions.*

Proof: Let us consider a special basis of maximally entangled states in $\mathbb{C}^d\otimes\mathbb{C}^d$ [31]:

$$|\Psi_{jk}\rangle = \frac{1}{\sqrt{d}}\sum_{n=1}^d \exp\Big(\frac{2\pi i}{d}jn\Big)|n, n\oplus k\rangle, \tag{30}$$

where $j,k = 1,\ldots,d$ and $\oplus$ means addition modulo d. Any equal weight combination of all states of the form (30), which belong to the same value of k, is then a projector onto a separable state since

$$\begin{aligned} P_k &= \sum_{j=1}^d |\Psi_{jk}\rangle\langle\Psi_{jk}| & (31)\\ &= \frac{1}{d}\sum_{j,n,m=1}^d \exp\Big[\frac{2\pi i}{d}j(n-m)\Big]|n, n\oplus k\rangle\langle m, m\oplus k| \\ &= \sum_{n=1}^d |n\rangle\langle n|\otimes|n\oplus k\rangle\langle n\oplus k| \end{aligned}$$

is an equal weight combination of product states. Here we have used that $\frac{1}{d}\sum_{j=1}^d \exp\Big[\frac{2\pi i}{d}j(n-m)\Big] = \delta_{n,m}$. Moreover, the reductions of the respective states P_k/d are maximally chaotic, i.e. $\rho_A = \mathbf{1}/d$, just as the reductions of any maximally entangled state.

If we now have a state with multiplicities being multiples of d we can replace the projectors onto its eigenspaces with sufficiently many projectors of the form P_k. The resulting state will then be again a convex combination of product states, i.e., separable, having the same spectrum and maximal chaotic reductions. $\square$

For the case of Werner states we note that the unitary invariance of the state $\rho(p)$ in Eq. (29) implies that its reductions are $\rho_A = \mathbf{1}/d$. Moreover $\rho(p)$ has two eigenvalues $(1-p)/r_+$ and p/r_- with multiplicities r_+, r_- which are indeed multiples of d in odd dimensions.

Following Proposition 2 we can now construct a state

$$\rho'(p) = \frac{(1-p)}{r_+}\sum_{k=1}^{r_+/d} P_k + \frac{p}{r_-}\sum_{l=1}^{r_-/d} P_{l+r_+/d}, \tag{32}$$

which has then both, the same spectrum and the same reductions as $\rho(p)$. However, as convex combination of separable states it is itself separable for any $0 \leq p \leq 1$.

V. CONCLUSION

We discussed conditional Rényi and Tsallis entropies and the relation between their positivity and other separability properties. We showed in particular that states having a negative conditional entropy are distillable since they violate the reduction criterion.

Conditional entropies are a special instance of criteria using just the spectra of a state and its reductions. Concerning the detection of entanglement, it was shown in Ref.[8] that majorization is the strongest spectral criterion, which uses the spectra of a state and just one of its reductions. Its relation to other separability criteria is yet not known. The present result and numerical evidence may indicate that majorization is also implied by the reduction criterion. However, the proof presented in Sec.III C does not work for arbitrary convex functions and in fact majorization is not implied by the conditional entropy criteria.

Concerning separability the most efficient criterion is still the PPT criterion, which is also a spectral criterion, however, for the partially transposed state. One interesting question in this context would therefore be: how can other (easy calculable) invariants provide information about the separability of a state, which is not yet encoded in the smallest eigenvalue of its partial transpose?

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